Stochastic Differential Equations

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This PDF is a "cheat sheet" of sorts, on Stochastic Differential Equations. All of the material here is from [Oks92].

§1. Brownian Motion

For any $x, y \in \mathbb{R}^n$, and $t \in \mathbb{R}_{>0}$, define

$$p(t,x,y) := (2\pi t)^{-n/2} \cdot \exp\left(-\frac{\|x-y\|_2^2}{2t}\right)$$

For $t = 0$, set $p(0,x,y)$ to be the Dirac Delta centred at x.
We can now define Brownian motion as follows: Brownian motion over $\mathbb{R}^n$ centered at 0 is a stochastic process $\{B_t\}_{t\in[0,\infty)}$, such that for *any* $0 \le t_1 \le t_2 \le \cdots \le t_n$, we have:

$$\Pr(B_{t_1} \in F_1, \ldots, B_{t_n} \in F_n) = \int_{F_1\times\cdots\times F_n} p(t_1,0,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t_n-t_{n-1},x_{n-1},x_n)dx_1\cdots dx_n \tag{1.1}$$

where $F_1, \ldots, F_n$ are arbitrary measurable subsets of $\mathbb{R}^n$.
Note that *a priori*, it is not clear that such a stochastic process even exists, just based on Eq. (1.1) alone. However, **Kolmogorov's extension theorem** guarantees that such a process exists.
A few properties of Brownian motion are as follows:

1. Observe that B_t is a Gaussian Random vector with mean 0 and covariance tI_n, i.e. $\mathbb{E}[B_t] = 0$, and $\mathbb{E}\left[B_t B_t^{\mathsf{T}}\right] = tI_n$, for all $t \ge 0$.

2. In general, for any $0 < t_1 \le t_2 \le \cdots \le t_k$, $Z := (B_{t_1}, B_{t_2}, \ldots, B_{t_k}) \in \mathbb{R}^{nk}$ is a Gaussian random vector in $\mathbb{R}^{nk}$.

3. $\mathbb{E}\left[B_s B_t^{\mathsf{T}}\right] = \min\{t,s\}\cdot I_n$, i.e. the covariance of B_s and B_t equals $\min\{t,s\}\cdot I_n$. Thus, B_t and B_s **are not independent** (otherwise their covariance would have been 0). We present the derivation for $n=1$, the calculation for higher dimensions is analogous. Also, assume $s \ge t$.

$$\mathbb{E}[B_s B_t] = \int_{\mathbb{R}^n\times\mathbb{R}^n} x_1 x_2 p(t,0,x_1)p(s-t,x_1,x_2)dx_1dx_2$$

Substitute $x_2 = x_1 + z$. Then the above integral equals

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} x_1(x_1+z)p(t,0,x_1)p(s-t,x_1,x_1+z)dx_1dz = \underbrace{\int_{\mathbb{R}^n\times\mathbb{R}^n} x_1^2 p(t,0,x_1)p(s-t,0,z)dx_1dz}_{=:I_1}$$
$$+ \underbrace{\int_{\mathbb{R}^n\times\mathbb{R}^n} x_1 z p(t,0,x_1)p(s-t,0,z)dx_1dz}_{=:I_2}$$

where we notice that $p(s-t,x_1,x_1+z) = p(s-t,0,z)$. Now,

$$I_1 = \int_{\mathbb{R}^n\times\mathbb{R}^n} x_1^2 p(t,0,x_1)p(s-t,0,z)dx_1dz = \int_{\mathbb{R}^n} x_1^2 p(t,0,x_1)dx_1 \cdot \int_{\mathbb{R}^n} p(s-t,0,z)dz = t\cdot 1 = t$$
$$I_2 = \int_{\mathbb{R}^n} x_1 p(t,0,x_1)dx_1 \cdot \int_{\mathbb{R}^n} zp(s-t,0,z)dz = 0\cdot 0 = 0$$

as desired.

4. For any $s,t,h,h' > 0$ such that $s \ge t+h'$, $B_{s+h} - B_s$ and $B_{t+h'} - B_t$ are independent. Indeed,

$$\begin{aligned}\mathbb{E}\left[(B_{s+h}-B_s)(B_{t+h'}-B_t)\right] &= \mathbb{E}[B_{s+h}B_{t+h'}] - \mathbb{E}[B_sB_{t+h'}] - \mathbb{E}[B_{s+h}B_t] + \mathbb{E}[B_sB_t] \\ &= t+h'-(t+h')-t+t = 0\end{aligned}$$

Since $B_{s+h} - B_s$ and $B_{t+h'} - B_t$ are both Gaussian RVs, and since for Gaussian RVs uncorrelatedness implies independence, we're done.

5. One can show that $\mathbb{E}\left[|B_t - B_s|^4\right] = n(n+2)|t-s|^2$. Consequently, by Kolmogorov's continuity theorem, there exists a stochastic process $\{\widetilde{B}_t\}_{t\in[0,\infty)}$ s.t. $\widetilde{B}_t = B_t$ a.e. for all $t \geq 0$, $\widetilde{B}_t$ satisfies Eq. (1.1), and $t \mapsto \widetilde{B}_t(\omega)$ is continuous for almost all $\omega \in \mathbb{R}$.
   Thus, WLOG we will assume that the Brownian motion we are working with is the continuous version, whose existence was outlined above.
6. Let $0 = t_1 < t_2 < \cdots < t_n = t$ be any partition $\mathcal{P}$ of $[0,t]$. Define $Y_{\mathcal{P}} := \sum_{k=1}^{n-1}(B_{t_{k+1}} - B_{t_k})^2$. Finally, let $\|\mathcal{P}\| := \max\{t_2 - t_1, \ldots, t_n - t_{n-1}\}$ be the "mesh" of $\mathcal{P}$. Then $\lim_{\|\mathcal{P}\|\to 0} Y_{\mathcal{P}} = t$. Consequently, it can be shown that the total variation of Brownian motion on $[0,t]$ is infinite.
7. Let $\mathcal{F}_s$ be the filtration generated by B_t for all $t \in [0,s]$. $\{B_t\}$ is a martingale w.r.t. this filtration, i.e. $\mathbb{E}\left[B_s|\mathcal{F}_t\right] = B_t$ for all $s \geq t$.

To quickly summarize the main points,

1. $\mathbb{E}\left[B_t\right] = 0$, and $\mathbb{E}\left[B_t B_t^\mathsf{T}\right] = tI_n$, for all $t \geq 0$.
2. For any $0 < t_1 \leq t_2 \leq \cdots \leq t_k$, $Z := (B_{t_1}, B_{t_2}, \ldots, B_{t_k}) \in \mathbb{R}^{nk}$ is a Gaussian random vector in $\mathbb{R}^{nk}$.
3. $\mathbb{E}\left[B_s B_t^\mathsf{T}\right] = \min\{t,s\} \cdot I_n$.
4. **Independent Increments**: For any $s, t, h, h' > 0$ such that $s \geq t + h'$, $B_{s+h} - B_s$ and $B_{t+h'} - B_t$ are independent.
5. $t \mapsto B_t(\omega)$ is continuous for almost all $\omega \in \mathbb{R}^n$. In fact, for every $\alpha < 1/2$, there exists a $C_\alpha > 0$ such that $|B_t(\omega) - B_s(\omega)| \leq C_\alpha |t-s|^\alpha$ for all $t \neq s$, and for almost all ω. Conversely, $\frac{|B_t(\omega) - B_s(\omega)|}{|t-s|^\alpha}$ is unbounded (as $t \to s$) for all $\alpha \geq 1/2$. In particular, $t \mapsto B_t(\omega)$ is differentiable nowhere.
6. $\lim_{\|\mathcal{P}\|\to 0} \sum_{k=1}^{n-1}(B_{t_{k+1}} - B_{t_k})^2 = t$. Consequently, the total variation of Brownian motion on $[0,t]$ is infinite.
7. Brownian motion is a martingale w.r.t. the filtration it generates.
8. $\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \ln\ln t}} = 1$ almost surely.

§2. Itô Integral

We give meaning to the integral $\int_S^T f(t,\omega) dB_t(\omega)$.
Let $f : [0,\infty) \times \Omega \mapsto \mathbb{R}$ be a measurable function satisfying the following properties:

1. $f(t,\omega)$ is $\mathcal{F}_t$-adapted, i.e. $\omega \mapsto f(t,\omega)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.
2. $\mathbb{E}\left[\int_S^T f(t,\omega)^2 dt\right] < \infty$ for almost all ω.

Then

$$\int_S^T f(t,\omega) dB_t(\omega) = \lim_{\|\mathcal{P}\|\to 0} \sum_j f(t_j,\omega) \mathbb{1}_{t\in[t_j,t_{j+1})}$$

Note that *unlike* the Riemann-Stieltjes integral, even if f is Itô-integrable, $\lim_{\|\mathcal{P}\|\to 0} \sum_j f(t_j^*,\omega) \mathbb{1}_{t\in[t_j,t_{j+1})}$ *may not* equal the Itô integral of f unless $t_j^* = t_j$ for all $j, \mathcal{P}$. In fact, when $t_j^* = \frac{t_j + t_{j+1}}{2}$, the integral we get is the *Stratonovich integral*, which does *not* equal the Itô integral in general, even when both of them exist.

Lemma 2.1 (Itô isometry)**.** If f is Itô-integrable,

$$\mathbb{E}\left[\left(\int_S^T f(t,\omega) dB_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_S^T f(t,\omega)^2 dt\right]$$

Some properties of the Itô integral:

1. $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$
2. $\int_S^T (\lambda f + \mu g) dB_t = \lambda \int_S^T f dB_t + \mu \int_S^T g dB_t$
3. $\mathbb{E}\left[\int_S^T f dB_t\right] = 0$
4. $\int_S^T f dB_t$ is $\mathcal{F}_T$-measurable
5. $t \mapsto \int_S^t f dB_t$ is a continuous function
6. $t \mapsto \int_S^t f dB_t$ is a martingale w.r.t. $\mathcal{F}_t$

§3. Itô's Formula

Let $g : [0, \infty) \times \mathbb{R} \mapsto \mathbb{R}$ be a doubly differentiable function. Let X_t be an Itô process [1]. Let $Y_t = g(t, X_t)$. Then

$$dY_t = (\partial_t g)dt + (\partial_x g)dX_t + \frac{1}{2}(\partial_x^2 g)(dX_t)^2$$

where $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$, and $(dB_t)^2 = dt$. Finally, $\int_0^t dZ_t = Z_t - Z_0$. Let's see some examples:

1. Let $X_t = B_t$, $g(t, x) = x^2/2$, i.e. $Y_t = B_t^2/2$. Then $dY_t = B_t dB_t + \frac{1}{2}dt$. Thus $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$. Notice the additional $t/2$ term.
2. Let $X_t = B_t$, $g(t, x) = tx$, i.e. $Y_t = tB_t$. Then $dY_t = B_t dt + t dB_t$. Thus $\int_0^t s dB_s = tB_t - \int_0^t B_s ds$. In general, let $c(s)$ be a differentiable function of s, with no dependence on t. Then $\int_0^t c(s) dB_s = c(t)B_t - \int_0^t B_s c'(s) ds$.

§4. Stochastic Differential Equations

Consider the differential equation

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)W_t$$

where $W_t \sim \mathcal{N}(0, 1)$ is "white noise". Then,

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

Let's see an example. Consider the stochastic version of the standard exponential-growth differential equation, i.e.

$$\frac{dN_t}{dt} = (r + \alpha W_t)N_t$$

Then

$$dN_t = rN_t dt + \alpha N_t dB_t \implies \frac{dN_t}{N_t} = r dt + \alpha dB_t \implies \int_0^t \frac{dN_t}{N_t} = rt + \alpha B_t$$

Now, apply Itô's formula with $g(t, x) = \ln x$, $X_t = N_t$. Then

$$d(\ln N_t) = \frac{1}{N_t} \cdot dN_t + \frac{1}{2} \cdot \left(-\frac{1}{N_t^2}(dN_t)^2\right)$$

Now,

$$(dN_t)^2 = (rN_t dt + \alpha N_t dB_t)^2 = r^2 N_t^2 (dt)^2 + 2r\alpha N_t^2 dt dB_t + \alpha^2 N_t^2 (dB_t)^2 = \alpha^2 N_t^2 dt$$

[1] we won't define what an Itô process is, since most "nice" functions satisfy the bill

Thus

$$d(\ln N_t) = \frac{1}{N_t} \cdot dN_t - \frac{\alpha^2}{2} dt \implies \int_0^t \frac{dN_t}{N_t} = \ln \frac{N_t}{N_0} + \frac{\alpha^2 t}{2}$$

and thus

$$N_t = N_0 \exp\left(\left(r - \frac{1}{2}\alpha^2\right) t + \alpha B_t\right)$$

Using the fact that $\mathbb{E}\left[\exp(\alpha B_t)\right] = 1$, we get that $\mathbb{E}\left[N_t\right] = \mathbb{E}\left[N_0\right] e^{rt}$, which matches our intuition.

References

[Oks92] Bernt Oksendal. *Stochastic Differential Equations (3rd Ed.): An Introduction with Applications*. Springer-Verlag, Berlin, Heidelberg, 1992.