Cheeger, Arora-Rao-Vazirani, and Buser: The Saga of the Sparsest Cut

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Roadmap

1. Sparsest Cut Problem
2. Cheeger's Inequality
3. Arora-Rao-Vazirani Algorithm
4. Buser's Inequality: Upgrading Cheeger for Cayley Graphs

The Second Eigenvalue

- For this talk, we will assume all graphs are unweighted, d-regular and connected.
- Define A as the adjacency matrix, then as usual define the Laplacian $L = I - \frac{A}{d}$.
- Call the eigenvalues as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n < 2$.
- Recall the all ones vector $\mathbf{1}$ is the vector with e-val 0.
- Variational definition of λ_2:

$$\lambda_2 = \min_{\langle x, \mathbf{1} \rangle = 0,\, x \neq 0} R(x)$$

where

$$R(x) = \frac{x^T L x}{x^T x}$$

Sparsest Cut Problem

- Sparsity/Conductance of a Cut $(S, \overline{S})$:

$$\phi(S) = \frac{e(S, \overline{S})}{d|S|}$$

- Conductance of a Graph

$$\phi(G) = \min_{|S| \leqslant n/2} \phi(S)$$

- **Sparsest Cut Problem:** Find cut witnessing conductance.

Spectral View of Sparsest Cut

- $\phi(G)$ can be seen as a "combinatorial analogue" of λ_2.
- If instead you only allow 0-1 vectors (i.e. integral cuts), then:

$$\mathbf{1}_S^T L \mathbf{1}_S = \frac{e(S, \overline{S})}{d}$$

- And one can show if one orthogonalizes $y_S = \frac{\mathbf{1}_S}{|S|} - \frac{\mathbf{1}}{n}$ that

$$R(y_S) = \frac{\phi(S)}{|\overline{S}|/n}$$

- Which means, if $|S| \leqslant n/2$ and $\phi(S) = \phi(G)$, that

$$\lambda_2 \leq R(y_S) \leq 2\phi(G)$$

Cheeger's Inequality

- Cheeger's Inequality [AM85]:

$$\frac{1}{2}\lambda_2 \leq \phi(G) \leq \sqrt{2\lambda_2}$$

- Algorithmic Proof idea: Show that there exists a randomized algorithm that gives a cut with expected conductance $\phi(S) \leq \sqrt{2\lambda_2}$.
- Randomized algorithm [Spi19], though trivially derandomized
  1. Find 2nd eigenvector, y.
  2. Relabel the vertices such that $y_1 \leq y_2 \leq \cdots \leq y_n$.
  3. Pick $i \in [n]$ u.a.r and output $S = [i]$.
- $\implies \lambda_2$ gives $\sqrt{1/\phi(G)}$-approx for sparsest cut.

Arora-Rao-Vazirani Algorithm [ARV09]

- Relax Sparsest Cut Problem to the following quantity, which is a 2-approximation

$$\psi(G) := \min_S \frac{e(S, \overline{S})}{\frac{d}{n}|S||\overline{S}|}$$

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- Relax further to rational function (can be minimized by SDP)

$$\text{minimize} \quad \frac{\sum_{\{i,j\} \in E} \|x_i - x_j\|^2}{\frac{d}{n} \sum_{i,j} \|x_i - x_j\|^2}$$

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- Tighten the relaxation by adding triangle inequalities, which would hold if x_i's were 0-1 indicator of a cut.

$$\text{subject to} \quad \begin{array}{ll} x_i \in \mathbb{R}^n & \forall i \in [n] \\ \|x_i - x_j\|^2 + \|x_j - x_k\|^2 \geqslant \|x_i - x_k\|^2 & \forall i,j,k \in [n] \end{array}$$

Arora-Rao-Vazirani Algorithm [ARV09]

- Property of the relaxation:

$$\lambda_2 \leq \mathrm{ARV}(G) \leq \psi(G)$$

- In their paper, ARV find a rounding scheme which achieves $O(\sqrt{\log n})$-approximation to $\mathrm{ARV}(G)$.
- Idea: Use $g \sim \mathcal{N}(0, I)$ and $y_i = \langle x_i, g \rangle$ to construct a linear embedding. Then round using thresholds of that embedding.

Cheeger's Inequality on Riemannian Manifolds

- Cheeger proved $\phi(M) \leq O(\sqrt{\lambda_2})$
- However, $\phi(M) \geq \frac{\lambda_2}{2}$ doesn't hold
  - Indicators of sets are not smooth
  - If you try to smooth them out, then $\langle f, Lf \rangle = \int_M \|\nabla f\|^2$ is big on the boundary; and M could have a lot of volume in this region

Buser's Inequality

- It turns out, the problem is manifolds with negative curvature.
- Buser was able to upgrade Cheeger to prove that for manifolds with nonnegative curvature,

$$\lambda_2 \leq 10\phi(M)^2$$

- Much stronger! Implies $\phi(M) = \Theta(\sqrt{\lambda_2})$

Buser's Holds in the Graph Case

It turns out that for some notions of curvature, **Abelian Cayley Graphs** have nonnegative curvature.

Definition

An Abelian Cayley Graph is given by an abelian group G and symmetric multiset of group elements $S = S^{-1}$. There is a vertex for all $g \in G$ and edges $(g, g + s)$ for all $s \in S$. We will identify some set of "positive elements" $S_+ \subseteq S$.

Theorem ([GT21])

Let G be a d-regular abelian Cayley graph. Then

$$\lambda_2(G) \leqslant O(d) \cdot \mathrm{ARV}(G)^2 \leqslant O(d) \cdot \psi(G)^2 \leqslant O(d) \cdot \phi(G)^2$$

- In particular, $\sqrt{\lambda_2}$ is a $O(\sqrt{d})$ approximation to $\phi(G)$.

Proof of Buser's Inequality For the Graph Case

- Let G^t be the graph with adjacency matrix A^t. Note that $\lambda_2(G^t) = 1 - (1 - \lambda_2(G))^t$.
- Then, the first inequality can be shown by proving that $\mathrm{ARV}(G^t) \leqslant \sqrt{2dt} \cdot \mathrm{ARV}(G)$ and choosing $t = 1/\lambda_2$

$$
\begin{aligned}
1 - \frac{1}{e} &\leqslant 1 - (1 - \lambda_2(G))^t = \lambda_2(G^t) \\
&\leqslant \mathrm{ARV}(G^t) \leqslant \sqrt{2dt} \cdot \mathrm{ARV}(G) = \sqrt{\frac{2d}{\lambda_2}} \cdot \mathrm{ARV}(G) \\
&\implies \lambda_2 \leqslant O(d) \cdot \mathrm{ARV}(G)^2
\end{aligned}
$$

Proof of Buser's Inequality For the Graph Case

- Note that we can write the ratio of quadratic forms as, for PSD X:
$$\frac{\sum_{\{i,j\}\in E}\|x_i - x_j\|^2}{\frac{d}{n}\sum_{i,j}\|x_i - x_j\|^2} = \frac{\langle L, X\rangle}{\langle L_n, X\rangle}$$
- If we permute the rows/cols as $(X_g)_{ij} = \langle x_{g+i}, x_{g+j}\rangle$ the objective doesn't change, so we can always replace $X \leftarrow \frac{1}{|G|}\sum_g X_g$.
- Then, by symmetry, for $s \in S$, $\|x_u - x_{u+s}\| = \|x_v - x_{v+s}\|$
- Thus, the objective becomes

$$\frac{\langle L, X\rangle}{\langle L_n, X\rangle} = \frac{\frac{n}{2}\sum_{s\in S}\|x_s - x_0\|^2}{\frac{d}{n}\sum_{i,j\in G}\|x_i - x_j\|^2} = \frac{n^2}{2}\cdot\frac{\mathbb{E}_{s\sim S}\|x_s - x_0\|^2}{\sum_{i,j\in G}\|x_i - x_j\|^2}$$

Proof of Buser's Inequality For the Graph Case

- It suffices to show that

$$\sqrt{2dt} \cdot \mathop{\mathbb{E}}_{s \sim S} \|x_s - x_0\|^2 \geqslant \mathop{\mathbb{E}}_{s_1,\ldots,s_t \sim S} \|x_{s_1+\cdots+s_t} - x_0\|^2$$

- Let $s_1 + \ldots + s_t = \sum_{s \in S_+} c_s \cdot s$ be the "reduced" version
- By squared triangle inequality,

$$\|x_{s_1+\cdots+s_t} - x_0\|^2 = \left\|x_{\sum_{s \in S_+} c_s \cdot s} - x_0\right\|^2 \leqslant \sum_{s \in S_+} c_s \cdot \|x_s - x_0\|^2$$

- Note that c_s is a sum of t independent $\{0, \pm 1\}$-valued random variables (which we call $X_1, \ldots, X_t$) with mean 0 which take the value 0 with probability $1 - 2/d$.

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- Note that c_s is a sum of t independent $\{0, \pm 1\}$-valued random variables (which we call $X_1, \ldots, X_t$) with mean 0 which take the value 0 with probability $1 - 2/d$.
- $\mathbb{E}\, c_s^2 = t\, \mathbb{E}\, X_1^2 = 2t/d$ and thus by Cauchy-Schwarz $\mathbb{E}\,|c_s| \leqslant \sqrt{2t/d}$.
- Therefore,

$$\mathbb{E} \left\| x_{\sum_{s \in S_+} c_s \cdot s} - x_0 \right\|^2 \leqslant \sqrt{2t/d} \cdot \sum_{s \in S_+} \mathbb{E} \|x_s - x_0\|^2 \leqslant \sqrt{2td} \cdot \mathbb{E}_{s \sim S} \|x_s - x_0\|^2$$

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